The following is the comparison of observed with computed positions :—

	Obs.	Comp.	O – C.	Obs.	Comp.	O – C.	
	°	°	°	″	″	″	
1881·71	124·1	123·1	+ 1·0	0·62	0·47	+·15	β 3
1888·17	105·5	106·5	– 1·0	0·49	0·54	–·05	Com. 3
1893·13	91·1	90·9	+ 0·2	0·25	0·41	–·16	Com. 1
1902·62	345·5	356·2	– 10·7	0·45	0·25	+·20	β 1
1905·68	331·4	327·6	+ 3·8	0·28	0·34	–·06	β 2
1906·72	326·7	322·2	+ 4·5	0·23	0·36	–·13	β 1
1907·69	316·5	317·2	– 0·7	0·31	0·38	–·07	β 1
1908·03	297·9	315·4	– 17·5	0·65	0·39	+·26	Gr. Obs. 1
1909·22	306·3	310·0	– 3·7	0·26	0·40	–·14	Fox 2
1910·78	301·9	303·0	– 1·1	0·26	0·40	–·14	Doo. 2
1911·59	295·4	299·4	– 4·0	0·29	0·38	–·09	Fox 5
1912·59	289·6	294·1	– 4·5	0·38	0·35	+·03	Fox 3
1913·71	290·2	286·3	+ 3·9	0·38	0·29	+·09	Fox 3
1914·64	281·7	276·2	+ 5·5	0·39	0·23	+·16	Fox 1

The hypothetical parallax is ″·0335.

Elimination of the known Integrals from the n-Body Problem.

By Professor L. Becker, Ph.D.

1. In the n-body-problem the $3n$ independent co-ordinates are referred to a fixed system. Six integrals belong to the motion of the centroid and can be eliminated by the introduction of $3(n-1)$ Jacobi co-ordinates, in which the system of axes moves with the centroid parallel to itself. Other three integrals give the position of the invariable plane and the angular momentum of the system. In this paper I shall introduce $3(n-1)$ new co-ordinates, and as many quantities which are canonically conjugated to them, and they are so chosen that four of them disappear from the characteristic function. Three of these four quantities have constant values, and express the same result as is contained in the three integrals mentioned. The latter are thus eliminated from the problem, and the problem is reduced to the order $6(n-1)-4=(6n-10)$. The problem has been solved before, though for different co-ordinates and by different methods.*

In my method I use for the relative position of the masses the plane passing through the three masses m_1, m_2, m_3 as plane of reference, with the line m_1m_2 as zero line, and I employ the three

* As to references, see E. T. Whittaker, "Prinzipien der Stoerungs-theorie . . ." in the *Encyklopaedie der mathematischen Wissenschaften*, 6, 2, 12, pp. 516, 521.

53

Eulerian angles for defining the position of the reference system. I finally eliminate Euler's angles and their conjugated quantities by the same contact transformation as is applied in the problem of rotation.

2. *Characteristic Function expressed in Relative Polar Coordinates and Euler's Angles.*—Let X, Y, Z designate Jacobi's co-ordinates, m_2 being referred to m_1, m_3 to the centroid, G_1, of

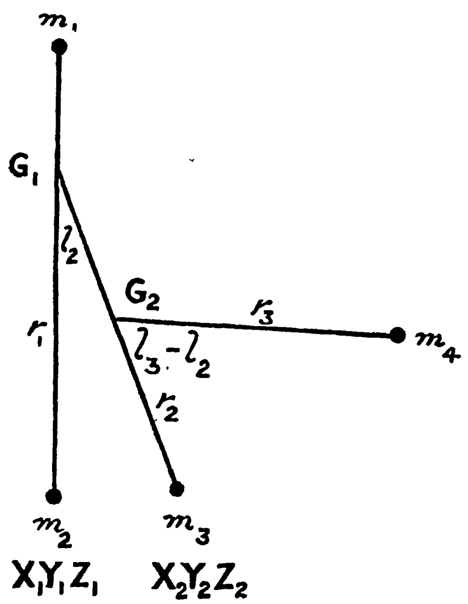

FIG. 1.

m_1 and m_2, m_4 to the centroid of m_1, m_2, m_3, etc. (see fig. 1). The kinetic energy of the system then is given by

$$T = \sum_1^{n-1} \frac{\mu_k}{2}[\dot{X}_k^2 + \dot{Y}_k^2 + \dot{Z}_k^2],$$

where μ_k are the well-known functions of the masses.

I introduce Euler's angles ψ, θ, ϕ, and the relative polar coordinates, viz. the distance, r_k; the longitude, l_k; and latitude, b_k. They are defined in section 1 and are shown in figs. 1 and 2. The total number of variables is again $3(n-1)$, since l_1, b_1, and b_2 have zero values by definition.

$$X = r\{\cos\psi\cos(\phi + l)\cos b + \sin\psi\cos\theta\sin(\phi + l)\cos b + \sin\psi\sin\theta\sin b\}$$

. . .

T becomes

$$T(r\dot{r}\ldots) = \sum_1^{n-1}\frac{\mu}{2}\{\dot{r}^2 + r^2[(\dot{\phi} + \dot{l})\cos b - \dot{\psi}(\cos\theta\cos b + \sin\theta\sin b\sin\phi + l) + \dot{\theta}\cos(\phi + l)\sin b]^2 + r^2[\dot{\psi}\cos(\phi + l)\sin\theta + \dot{\theta}\sin(\phi + l) - \dot{b}]^2\}.$$

The suffixes, k, have been omitted.

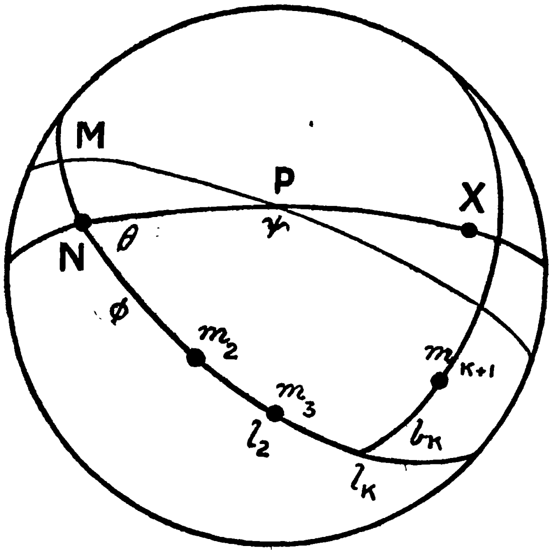

FIG. 2.

Use capital letters to denote the quantities which are canonically conjugated to the co-ordinates, *e.g.* $L_k = \partial T(r\dot{r} \ldots)/\partial l_k$, then the characteristic function, $H = T - U$, becomes

$$
\begin{aligned}
H(R, r \ldots) = \frac{1}{2\mu_1} \Big\{ & R_1^2 + r_1^{-2}\Big(\Phi - L_2 - \sum_3^{n-1} L\Big)^2 \\
& + (r_1 \sin l_2)^{-2}\Big[F_1 + \sum_3^{n-1}(L \tan b \cos (l - l_2) - B \sin (l - l_2))\Big]^2 \Big\} \\
(1) \qquad + \frac{1}{2\mu_2} \Big\{ & R_2^2 + r_2^{-2} L_2^2 \\
& + (r_2 \sin l_2)^{-2}\Big[F_2 + \sum_3^{n-1}(L \tan b \cos l - B \sin l)\Big]^2 \Big\} \\
+ \sum_3^{n-1} \frac{1}{2\mu} & [R^2 + (r \cos b)^{-2} L^2 + r^{-2} B^2] - f \sum \frac{m_a m_b}{r_{ab}}
\end{aligned}
$$

where

$$
\begin{aligned}
F_1 &= -\Theta \cos (\phi + l_2) + (\Psi + \Phi \cos \theta) \sin (\phi + l_2)/\sin \theta, \\
F_2 &= -\Theta \cos \phi \qquad\quad + (\Psi + \Phi \cos \theta) \sin \phi/\sin \theta.
\end{aligned}
$$

ψ and the time do not occur in H, and hence H and Ψ have constant values.

3. *Contact Transformation.*—The terms F_1 and F_2 are analogous to those in the characteristic function in the problem of rotation. As already mentioned, I make the same change of variables. Instead of the three Eulerian angles, ψ, θ, ϕ, I introduce the arcs shown in heavy lines in fig. 3.

According to the diagram the three independent variables can be replaced by not less than five quantities, *i.e.* the three arcs m_2M,

MP, PX, and the two angles at M and P. It can be proved that if the three arcs be chosen as the new co-ordinates (q_1, q_2, q_3), the angles at M and P are functions of the quantities "p" which are canonically conjugated to them. I apply the following transformation:

$$q_1 = \psi - a, \qquad \Psi = p_1$$

$$q_2 = q_2\,(\theta, \mathrm{A}, \mathrm{B}), \qquad \Theta = -p_2 \sin b \sin \mathrm{A}$$

$$q_3 = \phi + b, \qquad \Phi = p_3$$

$$\cos \mathrm{A} = -\frac{p_3}{p_2} \qquad \cos \mathrm{B} = +\frac{p_1}{p_2}$$

In these formulæ a, b, and q_2 are to be replaced by the values obtained from the spherical triangle (fig. 3) as functions of θ, p_1, p_2, p_3. That this is a contact transformation is evident from the following equations, which are the result of differentiation of the

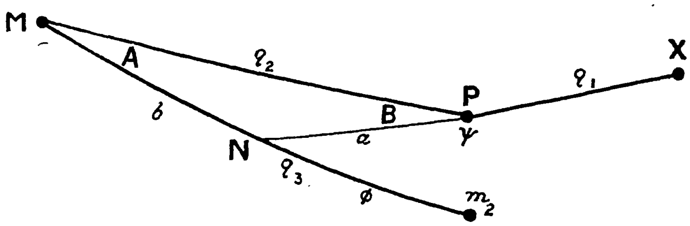

FIG. 3.

above equations. The quantities differentiated are supposed to be expressed as functions of ψ, θ, ϕ and p_1, p_2, p_3.

$$\frac{\partial q_3}{\partial \theta} = \frac{\partial \Theta}{\partial p_3} \qquad \frac{\partial q_3}{\partial p_2} = \frac{\partial q_2}{\partial p_3} \qquad \frac{\partial q_3}{\partial p_1} = \frac{\partial q_1}{\partial p_3}$$

$$\frac{\partial q_2}{\partial \theta} = \frac{\partial \Theta}{\partial p_2} \qquad \frac{\partial q_2}{\partial p_1} = \frac{\partial q_1}{\partial p_2} \qquad \frac{\partial q_1}{\partial \theta} = \frac{\partial \Theta}{\partial p_1}$$

$$\frac{\partial q_3}{\partial \phi} = \frac{\partial \Phi}{\partial p_3} \qquad \frac{\partial q_1}{\partial \psi} = \frac{\partial \Psi}{\partial p_1}$$

The other differential quotients are either zero or not required.

The transformation reduces the complicated expressions for F_1 and F_2 to the simple formulæ

$$(2)\quad \mathrm{F}_1 = (\alpha_3{}^2 - p_3{}^2)^{\frac{1}{2}} \sin (q_3 + l_2) \text{ and } \mathrm{F}_2 = (\alpha_3{}^2 - p_3{}^2)^{\frac{1}{2}} \sin q_3.$$

These and $\Phi = p_3$ (in first line) have to be substituted in (1). Neither the time nor p_1 nor q_1 nor q_2 occur in H (R, r, q, p), and hence $\mathrm{H} = \alpha_1$, $q_1 = \beta_2$, $p_1 = \alpha_2$ (as in section 2), $p_2 = \alpha_3$, and $\cos \mathrm{B} = \alpha_2/\alpha_3$ where α, β designate canonic constants. The value of p_2 $(= \alpha_3)$ is already entered in formula (2). MP (fig. 3) is therefore the invariable plane. H contains thus only q_3 and p_3 in addition to the $6n-12$ variables r_k, l_k, b_k, R_k, L_k, B_k, and the problem is of the order $6n-10$.

In the *3-body*-problem the sums $\sum_{3}^{n-1}$ in (1) become zero and the problem is of the eighth order; in the co-planar 3-body-problem the second and fourth lines in (1) disappear, as also all the q and p variables, since $\Phi = p_3$ now obtains a constant value. The co-planar 3-body-problem is thus of the sixth order.

Crieff:
1920 *July* 7.

Note on Equal-Area Maps. By F. Bowman, M.A.

(*Communicated by the Secretaries.*)

It is known that all those equal-area representations of the plane (X, Y) on the plane (x, y) in which the parallel straight lines $Y = \text{const.}$ correspond to the parallel straight lines $y = \text{const.}$ are given by

$$x = \frac{X}{\phi'(Y)} + \omega(Y),$$
$$y = \phi(Y),$$

where ϕ and ω are two arbitrary functions.*

The simplest equal-area representation of a sphere, known in principle to Archimedes, is that which is obtained by projecting the points of the sphere on to a tangent cylinder, the rays of projection passing through the axis of the cylinder at right angles, and then developing the cylinder into a plane. In the case of the earth (latitude u, longitude v), the equator being the great circle of contact between the earth and the cylinder, the equations of this projection are

$$X = v,$$
$$Y = \sin u.$$

It follows that all those equal-area maps of the earth in which the parallels of latitude $u = \text{const.}$ become the parallel straight lines $y = \text{const.}$ are included in the formulæ

$$x = \frac{v}{\phi'(\sin u)} + \omega(\sin u),$$
$$y = \phi(\sin u);$$

and these two equations may be replaced by

$$x = vf'(y) + w(y) \quad . \quad . \quad . \quad . \quad (1)$$

where f and w are arbitrary functions, together with the relation

$$f(y) = \sin u,$$

which enables us to graduate the y-axis in terms of u.

* *Vide* Scheffers, *Theorie der Curven*, p. 123; or Gravé, Liouville, 1896, p. 320.